Modulus, Argument, Polar Form, Argand diagram and deMoivre's Theorem

1. **(i)** $\dfrac{1+i}{1-i} = \dfrac{\sqrt{2}\text{cis}(\pi/4)}{\sqrt{2}\text{cis}(-\pi/4)} = \text{cis}\,\dfrac{\pi}{2}$, $\quad \left|\dfrac{1+i}{1-i}\right| = 1$, $\quad \text{Arg}\left(\dfrac{1+i}{1-i}\right) = \dfrac{\pi}{2}$

or $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{1+2i+i^2}{1-i^2} = \dfrac{2i}{2} = i$, $\quad \left|\dfrac{1+i}{1-i}\right| = 1$, $\quad \text{Arg}\left(\dfrac{1+i}{1-i}\right) = \dfrac{\pi}{2}$

(ii) $z = \dfrac{1+\sqrt{2}+i}{1-i} = \dfrac{(1+\sqrt{2}+i)(1+i)}{(1-i)(1+i)} = \dfrac{\sqrt{2}+(2+\sqrt{2})i}{2}$

$|z| = \sqrt{\left(\dfrac{\sqrt{2}}{2}\right)^2 + \left(\dfrac{2+\sqrt{2}}{2}\right)^2} = \sqrt{2+\sqrt{2}}$, $\quad \text{Arg}(z) = \tan^{-1}\dfrac{2+\sqrt{2}}{\sqrt{2}} = \dfrac{3\pi}{8} \approx 1.1781$

(iii) $z = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta)$, $\quad |z| = 1$, $\text{Arg}(z) = -\theta$

(iv) $z = 1 + i\tan\theta$, $\quad |z| = \sqrt{1+\tan^2\theta} = \sec\theta$, $\quad \text{Arg}(z) = \tan^{-1}(\tan\theta) = \theta$

2. Put $z_k = x_k + y_k i$.

3. $|z - 2 - i| < 2$ shows the interior of the circle

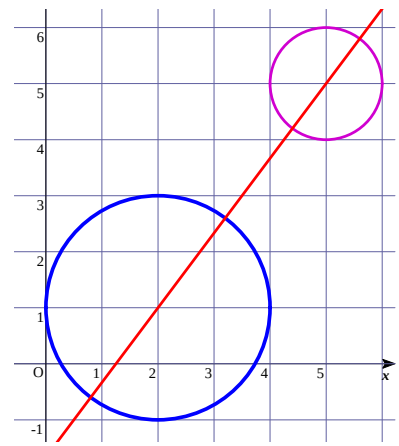
centre $C_1 = (2, 1)$, radius $r_1 = 2$

$|w - 5 - 5i| < 1$ shows the interior of the circle

centre $C_2 = (5, 5)$, radius $r_2 = 1$

Max of $|z - w| = \sqrt{(5-2)^2 + (5-1)^2} + 2 + 1 = 8$

Min of $|z - w| = \sqrt{(5-2)^2 + (5-1)^2} - 2 - 1 = 2$



4. Put $z_k = x_k + y_k i$, $k = 1, 2$.

$$w = \frac{z_1+z_2}{z_1-z_2} = \frac{(x_1+x_2)+(y_1+y_2)i}{(x_1-x_2)+(y_1-y_2)i} = \frac{[(x_1+x_2)+(y_1+y_2)i][(x_1-x_2)-(y_1-y_2)i]}{(x_1-x_2)^2+(y_1-y_2)^2}$$

$$\text{Re}(z) = \frac{(x_1+x_2)(x_1-x_2)+(y_1+y_2)(y_1-y_2)}{(x_1+x_2)^2+(y_1+y_2)^2} = 0 \iff x_1^{\,2} + y_1^{\,2} = x_2^{\,2} + y_2^{\,2} \iff |z_1| = |z_2|$$

5. $f(z) = \displaystyle\sum_{k=0}^{n} a_k z^k = \sum_{k=0}^{n} a_k (r\text{cis}\theta)^k = \sum_{k=0}^{n} r^k a_k \text{cis}(k\theta)$

$$\overline{f(z)} = f(\bar{z}) = \sum_{j=0}^{n} a_j \bar{z}^j = \sum_{j=0}^{n} a_j [r\text{cis}(-\theta)]^j = \sum_{j=0}^{n} r^j a_j \text{cis}(-j\theta)$$

$$|f(z)|^2 = f(z)\overline{f(z)} = \sum_{k=0}^{n} r^k a_k \text{cis}(k\theta) \sum_{j=0}^{n} r^j a_j \text{cis}(-j\theta) = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \text{cis}(k\theta)\text{cis}(j\theta) = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \text{cis}(k-j)\theta$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \left\{\frac{1}{2}[\text{cis}(k-j)\theta + \text{cis}(j-k)\theta]\right\} = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \left\{\frac{1}{2}[\text{cis}(k-j)\theta + \text{cis}(-(k-j)\theta)]\right\}$$

$$= \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \left\{\frac{1}{2}\left[\text{cis}(k-j)\theta + \overline{\text{cis}(k-j)\theta}\right]\right\} = \sum_{k=0}^{n}\sum_{j=0}^{n} r^{k+j} a_k a_j \cos(k-j)\theta$$

6. **(i)** Let $z_1 = |z_1| \text{ cis } \theta_1$, $z_2 = |z_2| \text{ cis } \theta_2$, then $\overline{z_2} = |z_2|\text{cis}(-\theta_2)$. If $z_1 z_2 \neq 0$,

$$\mathbf{R}(z_1\bar{z}_2) = |z_1||z_2| \Leftrightarrow \mathbf{R}(|z_1||z_2|\text{cis}(\theta_1 - \theta_2)) = |z_1||z_2| \Leftrightarrow |z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1||z_2|$$

$$\Leftrightarrow \cos(\theta_1 - \theta_2) = 1 \Leftrightarrow \theta_1 - \theta_2 = \pm 2n\pi \Leftrightarrow \arg z_2 = \arg z_1 \pm 2n\pi \text{ , where } n = 0, 1, 2, \ldots$$

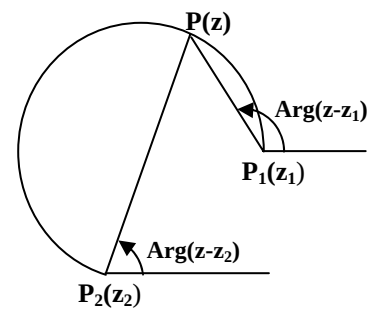
(ii) $|z_1 + z_2| = |z_1| + |z_2| \Leftrightarrow |z_1 + z_2|^2 = (|z_1| + |z_2|)^2 \Leftrightarrow |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$$\Leftrightarrow (z_1 + z_2)\overline{(z_1 + z_2)} = z_1\overline{z_1} + z_2\overline{z_2} + 2|z_1||z_2| \Leftrightarrow (z_1 + z_2)(\overline{z_1} + \overline{z_2}) = z_1\overline{z_1} + z_2\overline{z_2} + 2|z_1||z_2|$$

$$\Leftrightarrow z_1\overline{z_2} + \overline{z_1}z_2 = 2|z_1||z_2| \Leftrightarrow z_1\overline{z_2} + \overline{z_1\overline{z_2}} = 2|z_1||z_2|$$

$$\Leftrightarrow 2\mathbf{R}(z_1\bar{z}_2) = 2|z_1||z_2| \Leftrightarrow \mathbf{R}(z_1\bar{z}_2) = |z_1||z_2|$$ Result follows from (a).

7. **(i)** The locus is an arc of the circle standing on the chord with end points z_1 and z_2 such that an angle $P_1PP_2 = \frac{\pi}{6}$ is subtended by the chord at points on the arc.



(ii) One of the two branches of hyperbola with foci at z_1 and z_2 .

The difference between the distances from z_1 to the branch and z_2 to the branch is 3.

(iii) Put $z = x + yi$, $|z + 3i|^2 - |z - 3i|^2 = 12$ becomes $[x^2 + (y + 3)^2] - [x^2 + (y - 3)^2] = 12$

or $y = 2$ which is a horizontal straight line.

(iv) Put $z = x + yi$, $|z + 3i|^2 + |z - 3i|^2 = 90$ becomes $[x^2 + (y + 3)^2] + [x^2 + (y - 3)^2] = 90$
or $x^2 + y^2 = 6^2$, which is a circle with $(0, 0)$ as centre and 6 as radius.

8 – 10. Omitted. Bookwork or simple evaluations.

11. $\sqrt{3} - i = 2\text{cis}\left(-\frac{\pi}{6}\right)$, $\sqrt{3} + i = 2\text{cis}\left(\frac{\pi}{6}\right)$

$$\left(\sqrt{3} - i\right)^n + \left(\sqrt{3} + i\right)^n = \left[2\text{cis}\left(-\frac{\pi}{6}\right)\right]^n + \left[2\text{cis}\left(\frac{\pi}{6}\right)\right]^n = 2^n\left[\text{cis}\left(-\frac{n\pi}{6}\right) + \text{cis}\left(\frac{n\pi}{6}\right)\right] = 2^n\left(2\cos\frac{n\pi}{6}\right) = 2^{n+1}\cos\frac{n\pi}{6}$$

12. $\left(1 + i\sqrt{3}\right)^n = a_n + ib_n \Rightarrow \left(1 - i\sqrt{3}\right)^{n-1} = \overline{\left(1 + i\sqrt{3}\right)^{n-1}} = \overline{\left(1 + i\sqrt{3}\right)^{n-1}} = \overline{a_{n-1} + ib_{n-1}} = a_{n-1} - ib_{n-1}$

$$\left(1 + i\sqrt{3}\right)^n\left(1 - i\sqrt{3}\right)^{n-1} = (a_n + ib_n)(a_{n-1} - ib_{n-1})$$

But, $\left(1 + i\sqrt{3}\right)^n\left(1 - i\sqrt{3}\right)^{n-1} = \left[\left(1 + i\sqrt{3}\right)\left(1 - i\sqrt{3}\right)\right]^{n-1}\left(1 + i\sqrt{3}\right) = 4^{n-1}\left(1 + i\sqrt{3}\right) = 4^{n-1} + 4^{n-1}\sqrt{3}i$ (1)

and $(a_n + ib_n)(a_{n-1} - ib_{n-1}) = (a_{n-1}b_n - a_nb_{n-1}) + i(a_na_{n-1} + b_nb_{n-1})$ (2)

Result follows by comparing real and imaginary parts of (1) and (2).

13. **(i)** $(\cos\theta - i\sin\theta)^n = \left(\overline{\cos\theta + i\sin\theta}\right)^n = \overline{(\cos\theta + i\sin\theta)^n} = \overline{\cos n\theta + i\sin n\theta} = \cos n\theta - i\sin n\theta$

(ii) $\dfrac{(1-i\tan\theta)^n}{(1-i\tan\theta)^n} = \left(\dfrac{1-i\tan\theta}{1+i\tan\theta}\right)^n = \left(\dfrac{\cos\theta - i\sin\theta}{\cos\theta + i\cos\theta}\right)^n = \dfrac{(\cos\theta - i\sin\theta)^n}{(\cos\theta + i\sin\theta)^n} = \dfrac{\cos n\theta - i\sin n\theta}{\cos n\theta + i\sin n\theta} = \dfrac{1-\tan n\theta}{1+\tan n\theta}$

$\therefore$ $(1 - i\tan\theta)^n (1 + i\tan n\theta) = (1 + i\tan\theta)^n (1 - i\tan n\theta)$, by cross multiplication.

(iii) $(1 + i)^{2n} + (1 - i)^{2n} = \left(\sqrt{2}\,\text{cis}\dfrac{\pi}{4}\right)^{2n} + \left(\sqrt{2}\,\text{cis}\left(-\dfrac{\pi}{4}\right)\right)^{2n} = 2^n\,\text{cis}\dfrac{n\pi}{2} + 2^n\,\text{cis}\left(-\dfrac{n\pi}{2}\right) = 2^n\left[\text{cis}\dfrac{n\pi}{2} + \text{cis}\left(-\dfrac{n\pi}{2}\right)\right]$

$$= 2^n\left[2\cos\frac{n\pi}{2}\right] = 2^{n+1}\cos\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ 2^{n+1} & \text{if } n/2 \text{ is an even integer,} \\ -2^{n+1} & \text{if } n/2 \text{ is an odd integer.} \end{cases}$$

14. $$\left(\frac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n = \left(\frac{1+\cos\left(\frac{\pi}{2}-\theta\right) + i\sin\left(\frac{\pi}{2}-\theta\right)}{1+\cos\left(\frac{\pi}{2}-\theta\right) - i\sin\left(\frac{\pi}{2}-\theta\right)}\right)^n = \left[\frac{2\cos^2\frac{1}{2}\left(\frac{\pi}{2}-\theta\right) + 2i\sin\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)}{2\cos^2\frac{1}{2}\left(\frac{\pi}{2}-\theta\right) - 2i\sin\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\cos\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)}\right]^n$$

$$= \left[\frac{\text{cis}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}{\text{cis}\left[-\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]}\right]^n = \left[\text{cis}\left(\frac{\pi}{2}-\theta\right)\right]^n = \cos\left(\frac{n\pi}{2}-n\theta\right) + i\sin\left(\frac{n\pi}{2}-n\theta\right),$$ by de Moivre's Theorem.

15. $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \ldots (\cos n\theta + i\sin n\theta) = 1$

$(\text{cis}\,\theta)(\text{cis}\,\theta)^2 \ldots (\text{cis}\,\theta)^n = \text{cis}\,0$

$(\text{cis}\,\theta)^{1+2+\ldots+n} = \text{cis}\,2k\pi$, where k is an integer.

$(\text{cis}\,\theta)^{n(n+1)/2} = \text{cis}\,2k\pi$, $\text{cis}\left(\dfrac{n(n+1)}{2}\theta\right) = \text{cis}\,2k\pi$

$\therefore \theta = \dfrac{4k\pi}{n(n+1)}$, where k is an integer.

16. $\alpha, \beta = 1 \pm i = \sqrt{2}\,\text{cis}\left(\pm\dfrac{\pi}{4}\right)$,

$$\alpha^{4m} + \beta^{4m} = \left[\sqrt{2}\,\text{cis}\left(\frac{\pi}{4}\right)\right]^{4m} + \left[\sqrt{2}\,\text{cis}\left(-\frac{\pi}{4}\right)\right]^{4m} = 2^{2m}[\text{cis}(m\pi) + \text{cis}(-m\pi)] = 2^{2m}[2\cos(m\pi)] = 2^{2m+1}(-1)^m$$

17. $f(z) = az\bar{z} + bz + \bar{b}\bar{z} + c$

(i) $af(z) = a^2z\bar{z} + abz + a\bar{b}\bar{z} + ac = a^2z\bar{z} + abz + a\bar{b}\bar{z} + b\bar{b} - b\bar{b} + ac = (az + \bar{b})(a\bar{z} + b) + ac - |b|^2$

$= (az + \bar{b})\overline{(az + \bar{b})} + ac - |b|^2 = |az + \bar{b}|^2 + ac - |b|^2$

(ii) $f(z) \geq 0$ for all z $\Leftrightarrow$ $af(z) \geq 0$ for all z, since a is positive.

$\Leftrightarrow$ $|az + \bar{b}|^2 + ac - |b|^2 \geq 0$ for all z, by (i)

$\Leftrightarrow$ $ac - |b|^2 \geq 0$, since $|az + \bar{b}|^2 \geq 0$ for all z.

$\Leftrightarrow$ $|b|^2 \leq ac$

(iii) The equation $f(z) = 0$ has a solution $\Leftrightarrow$ $\exists z_0 \in \mathbf{C}$ s.t. $f(z_0) = 0$

$$\Leftrightarrow \left|az_0 + \overline{b}\right|^2 + ac - |b|^2 = 0 \quad \Leftrightarrow \quad \left|az_0 + \overline{b}\right|^2 = |b|^2 - ac \geq 0 \quad \Leftrightarrow \quad |b|^2 \geq ac$$

18. Bookwork : $|z_1 + z_2| \leq |z_1| + |z_2|$

$$\begin{aligned} |1 + a_1z + \ldots + a_Nz^N| &\leq |1| + |a_1z| + \ldots + |a_Nz^N| = 1 + |a_1||z| + \ldots + |a_N||z|^N \\ &< 1 + 2|z| + \ldots + 2|z|^N, \quad \text{since } |a_n| < 2 \text{ for } 1 \leq n \leq N \\ &= 2[|z| + \ldots + |z|^N] - 1 < 2\left[\frac{1}{3} + \ldots + \left(\frac{1}{3}\right)^n\right] - 1 = 2 \times \frac{1}{3} \times \frac{1-(1/3)^n}{1-(1/3)} - 1 = -\left(\frac{1}{3}\right)^n < 0 \end{aligned}$$

Contradict to the fact that z satisfies the equation $1 + a_1z + \ldots + a_Nz^N = 0$ and therefore

$|1 + a_1z + \ldots + a_Nz^N| = 0$

19. $|z^n \cos\theta_0 + z^{n-1} \cos\theta_1 + \ldots + \cos\theta_n| \leq \left|z^n \cos\theta_0\right| + \left|z^{n-1} \cos\theta_1\right| + \ldots + \left|\cos\theta_n\right|$

$$\leq |z|^n |\cos\theta_0| + |z|^{n-1} |\cos\theta_1| + \ldots + |\cos\theta_n| \leq |z|^n + |z|^{n-1} + \ldots + 1 \qquad \ldots..(1)$$

Suppose the given equation has a root inside the circle $|z| = \frac{1}{2}$, then $|z| < \frac{1}{2}$.

From (1), $|z^n \cos\theta_0 + z^{n-1} \cos\theta_1 + \ldots + \cos\theta_n|$

$$\leq |z|^n + |z|^{n-1} + \ldots + 1 \leq \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \ldots + 1 = \frac{1-(1/2)^n}{1-(1/2)} = 2\left[1 - \left(\frac{1}{2}\right)^n\right] < 2$$

Contradict to the fact that z satisfies the equation $z^n \cos\theta_0 + z^{n-1} \cos\theta_1 + \ldots + \cos\theta_n = 2$ and therefore $|z^n \cos\theta_0 + z^{n-1} \cos\theta_1 + \ldots + \cos\theta_n| = 2$.

20. **(i)** $|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)} = (z_1 + z_2)(\overline{z_1} + \overline{z_2}) + (z_1 - z_2)(\overline{z_1} - \overline{z_2})$

$= 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2|z_1|^2 + 2|z_2|^2$

(ii) We use Induction to show that $P(n) : |a_n|^2 + |b_n|^2 = 2^{n+1}$ for all integers $n \geq 0$

$P(0) : |a_0|^2 + |b_0|^2 = 1^2 + 1^2 = 2 = 2^{0+1}$, since $a_0 = b_0 = c = \cos\theta + i \sin\theta$

$\therefore$ $P(0)$ is true.

Assume $P(k)$ is true for some integer k, $|a_k|^2 + |b_k|^2 = 2^{k+1}$ $\ldots.(1)$

For $P(k+1)$, $|a_{k+1}|^2 + |b_{k+1}|^2$

$$= \frac{1}{2}\left(|a_{k+1} + b_{k+1}|^2 + |a_{k=1} - b_{k+1}|^2\right) = \frac{1}{2}\left(\left|\left(a_k + c^{2^k} b_k\right) + \left(a_k - c^{2^k} b_k\right)\right|^2 + \left|\left(a_k + c^{2^k} b_k\right) - \left(a_k - c^{2^k} b_k\right)\right|^2\right)$$

$$= 2\left(|a_k|^2 + \left|c^{2^k} b_k\right|^2\right) = 2\left(|a_k|^2 + \left|c^{2^k}\right| |b_k|^2\right) = 2\left(|a_k|^2 + |c|^{2^k} |b_k|^2\right) = 2\left(|a_k|^2 + 1^{2^k} |b_k|^2\right) = 2\left(|a_k|^2 + |b_k|^2\right)$$

$= 2\ 2^{k+1}$, by (1)

$= 2^{(k+1)+1}$. $\therefore$ $P(k+1)$ is also true.

By the Principle of Mathematical Induction, $P(n)$ is true $\forall\ n \in \mathbf{N} \cup \{0\}$.

Obviously, $|a_n|^2 \leq |a_n|^2 + |b_n|^2 = 2^{n+1}$ and $|b_n|^2 \leq |a_n|^2 + |b_n|^2 = 2^{n+1}$

$\therefore$ $|a_n| \leq \left(\sqrt{2}\right)^{n+1}$ and $|b_n| \leq \left(\sqrt{2}\right)^{n+1}$

21. **(i)** $|z_1+z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}=(z_1+z_2)(\overline{z_1}+\overline{z_2})=|z_1|^2+|z_2|^2+z_1\overline{z_2}+z_2\overline{z_1}$

$=|z_1|^2+|z_2|^2+z_1\overline{z_2}+\overline{z_1\overline{z_2}}=|z_1|^2+|z_2|^2+2\mathbf{R}\left(z_1\overline{z_2}\right)\le|z_1|^2+|z_2|^2+2|z_1|\left|\overline{z_2}\right|=|z_1|^2+|z_2|^2+2|z_1||z_2|$

$=|z_1|^2+|z_2|^2+2\sqrt{\left(c|z_1|^2\right)\left(\frac{1}{c}|z_2|^2\right)}\le|z_1|^2+|z_2|^2+c|z_1|^2+\frac{1}{c}|z_2|^2$, by G.M. $\le$ A.M.

$=(1+c)\,|z_1|^2+(1+c^{-1})\,|z_2|^2$

The equality holds when $\mathbf{R}\left(z_1\overline{z_2}\right)=|z_1|\left|\overline{z_2}\right|$ and $c|z_1|^2=\frac{1}{c}|z_2|^2$

$\Leftrightarrow$ $z_1\overline{z_2}$ is real and $|z_1|=c|z_2|$, since c is positive

$\Leftrightarrow$ $\arg(z_1\overline{z_2})=0$ or π and $|z_1|=c|z_2|$

$\Leftrightarrow$ $\arg(z_1)+\arg(\overline{z_2})=0$ or π and $|z_1|=c|z_2|$

$\Leftrightarrow$ $\arg(z_1)-\arg z_2=0$ or π and $|z_1|=c|z_2|$

$\Leftrightarrow$ $\arg(z_1)=\arg z_2+\ 0$ or π and $|z_1|=c|z_2|$

$\Leftrightarrow$ $z_1=\pm cz_2$

(ii) Let P(n) be the proposition : if the a's are positive numbers such that $a_1^{-1}+\ldots+a_n^{-1}=1$, then $|z_1+\ldots+z_n|^2\le a_1|z_1|^2+\ldots+a_n|z_n|^2$

P(1) is obviously true.

Assume P(k) is true for some $k\in\mathbf{N}$. i.e.

$a_1^{-1}+\ldots+a_k^{-1}=1,\quad a_i>0,\quad |z_1+\ldots+z_k|^2\le a_1|z_1|^2+\ldots+a_k|z_k|^2$ …..(1)

For P(k+1),

Let $a_1^{-1}+\ldots+a_k^{-1}+a_{k+1}^{-1}=1,\quad a_i>0,$

Put $b_k^{-1}=a_k^{-1}+a_{k+1}^{-1}$, then $b_k=\dfrac{a_ka_{k+1}}{a_k+a_{k+1}}$ and $a_1^{-1}+\ldots+a_{k-1}^{-1}+b_k^{-1}=1,$

By (1) we get

$|z_1+\ldots+z_k+z_{k+1}|^2\le a_1|z_1|^2+\ldots+a_{k-1}|z_{k-1}|^2+b_k|z_k+z_{k+1}|^2$ …..(2)

But $b_k|z_k+z_{k+1}|^2\le b_k\,[(1+c)\,|z_k|^2+(1+c^{-1})\,|z_{k+1}|^2]$, by **(i)**

$\le\dfrac{a_ka_{k+1}}{a_k+a_{k+1}}\left[\left(1+\dfrac{a_k}{a_{k+1}}\right)\right]|z_k|^2+\left[\left(1+\dfrac{a_{k+1}}{a_k}\right)\right]|z_{k+1}|^2$, by putting $c=\dfrac{a_k}{a_{k+1}}$.

$\le a_k|z_k|^2+a_{k+1}|z_{k+1}|^2$ …..(3)

Substitute (3) in (2), $|z_1+\ldots+z_k+z_{k+1}|^2\le a_1|z_1|^2+\ldots+a_{k-1}|z_{k-1}|^2+a_k|z_k|^2+a_{k+1}|z_{k+1}|^2$.

$\therefore$ P(k+1) is also true.

By the Principle of Mathematical Induction, P(n) is true $\forall\ n\in\mathbf{N}$.